Sensitivity Controller for Uncertain Systems

Kenji Okada*
Mitsubishi Electric Corporation, Kanagawa, Japan
and
Robert E. Skelton†
Purdue University, West Lafayette, Indiana 47906

In this paper a new controller design, which we shall call the "trajectory sensitivity optimization" method, is presented to improve the robustness for parameter variations. The method uses the sensitivity trajectory to model the parameter uncertainty and introduces a special quadratic cost function involving an input and output sensitivity term. Necessary conditions are derived to obtain the dynamic controller. The necessary conditions consist of two Lyapunov equations and two controller gain equations that have no closed-form solution. Therefore, a special iterative algorithm was developed to obtain the numerical solution. The method can deal with a wider class of parameter uncertainty than existing methods. Numerical examples show that the method is effective in improving the robustness to parameter variations.

I. Introduction

THE linear quadratic Gaussian (LQG) theory is well established in multivariable control design synthesis, but it suffers from a poor sensitivity to certain classes of plant parameter uncertainty.[1] This sensitivity problem for parameter uncertainty becomes extremely important in flexible structure control where there is large parameter uncertainty. To cope with the problem, various design syntheses have been proposed.

We are motivated by the trajectory sensitivity approach of Yedavalli and Skelton[13] where the necessary conditions are described to solve our problem. By restricting our attention to controllers of order n (equal to plant order), we will be able to make further progress toward solutions.

The sensitivity controller proposed by Wagie and Skelton[2] uses a trajectory sensitivity model to include the effects of parameter uncertainty and a special cost function involving both an output and input sensitivity term. This paper shows how to reduce the sensitivity model to tractable order, while preserving the correlations between outputs and all of their sensitivities. The main drawback of this sensitivity controller design method is that the method does not deal with parameter uncertainty in the measurement matrix.

The maximum entropy method has been applied by Bernstein and Hyland[3] to the flexible structure problems. This method uses stochastic modeling for the parameter uncertainty to improve the parameter variation robustness. The design synthesis provides a direct method for the design of robust, reduced-order controllers in which robust controller design and controller order reduction are performed simultaneously. The necessary conditions obtained by this method consist of two modified Riccati equations and two modified Lyapunov equations coupled by stochastic effects. Two restrictions of the method relate to the structure of the parameter uncertainties permitted. The uncertain parameters must appear linearly in the plant input and output matrices. It also requires that the control- and measurement-dependent uncertain parameters are uncorrelated. Because of this requirement, the method cannot be applied directly to the problems in which there exists parameter uncertainty that affects the control matrix and the measurement matrix simultaneously. The method also cannot deal with parameter uncertainties in the disturbance matrix and in the output matrix. This may cause the unnecessary degradation of the closed-loop system performance.

The approach developed by Tahk and Speyer[4] is called asymptotic LQG design synthesis. This method uses the internal feedback loop to model the parameter variations and serves to improve the stability robustness and reduce the sensitivity to parameter variation. This approach is a generalization of the LQG/LTR technique introduced by Doyle and Stein.[5] The approach has difficulties when there exist parameter variations in the input matrix B or in the measurement matrix M. In this case, the method requires augmentation of the state space so that ΔB and ΔM are embedded in the state matrix of the augmented system. This augmentation of the state space eventually leads to the increase of controller order.

As explained so far, the existing robust controller design methods for parameter uncertainty have some restrictions on the structure of parameter uncertainty. Hence, the main purpose of this paper is to propose a new robust controller design synthesis that can deal with wider classes of parameter uncertainty. The proposed method uses trajectory sensitivity to model the parameter uncertainty and introduces the special cost function that includes the output and input sensitivity terms in addition to the nominal input and output cost. The controller parameters are determined such that the given cost function is minimized. Through this minimization procedure, the controller obtains a robustness property with respect to parameter variation. The fundamental idea of this method is the same as the Wagie-Skelton method, although the approach to obtaining the controller is different.

This paper is organized as follows. Section II discusses the modeling of parameter uncertainty using a trajectory sensitivity model. Section III introduces the newly developed "trajectory sensitivity optimization" method and provides the necessary conditions for the sensitivity-reducing controller and the algorithm to obtain the solution. Section IV deals with the numerical examples to demonstrate the effectiveness of the proposed method and provides performance comparisons with other design methods. Finally, Sec. V contains the conclusions.

Received Jan. 21, 1988; revision received Sept. 30, 1988.

*Control Engineering Section, Development and Design Department, Kamakura Works.

†Professor, Deparment of Aeronautics and Astronautics. Associate Fellow AIAA.

II. Modeling of Parameter Uncertainty

Trajectory Sensitivity Model

In this section, we derive the trajectory sensitivity model for a simply supported beam example that will be used in the numerical example in Sec. IV.

Assume that there are h uncertain parameteres $p_1, p_2, \ldots, p_h$ and a space-state model of the system given by

$$\dot{x} = A(p)x + B(p)u + D(p)w \tag{1a}$$

$$y = C(p)x \tag{1b}$$

$$z = M(p)x + v \tag{1c}$$

where x, y, z, u, w, and v are, respectively, state vector of dimension k, measurement vector of dimension l, input vector of dimension m, zero-mean white noise of dimension d with intensity W, and zero-mean white noise of dimension l with intensity V, and p is given by $p = [p_1, p_2, \ldots, p_h]^T$.

The resulting sensitivity system can be expressed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_s = A_s x_s + B_s u_s + D_s w_s \tag{2a}$$

$$y_s = C_s x_s \tag{2b}$$

$$z_s = M_s x_s + v_s \tag{2c}$$

where

$$x_s = \begin{Bmatrix} x \\ x_p \end{Bmatrix}, \quad y_s = \begin{Bmatrix} y \\ y_p \end{Bmatrix}, \quad z_s = \begin{Bmatrix} z \\ z_p \end{Bmatrix}$$

$$u_s = \begin{Bmatrix} u \\ u_p \end{Bmatrix}, \quad w_s = w, \quad v_s = \begin{Bmatrix} v \\ 0 \end{Bmatrix}$$

$$A_s = \begin{bmatrix} A & 0 \\ A_p & \tilde{A} \end{bmatrix}, \quad B_s = \begin{bmatrix} B & 0 \\ B_p & \tilde{B} \end{bmatrix}$$

$$C_s = \begin{bmatrix} C & 0 \\ C_p & \tilde{C} \end{bmatrix}, \quad D_s = \begin{bmatrix} D \\ D_p \end{bmatrix}, \quad M_s = \begin{bmatrix} M & 0 \\ M_p & \tilde{M} \end{bmatrix}$$

$$[\tilde{\cdot}] \triangleq \text{block diag } \{[\cdot], \cdots, [\cdot]\}$$

$$[\cdot]_p \triangleq \left[\frac{\partial [\cdot]^T}{\partial p_1} \cdots \frac{\partial [\cdot]^T}{\partial p_h} \right]^T$$

where the matrices A_s, B_s, C_s, D_s, and M_s are evaluated at $p = \bar{p}$ (nominal vector value of p). The basic idea to improve the robustness for parameter variations is to use a cost function V_D given by

$$V_D = E_\infty [y^T Q y + u^T R u + \sum_{i=1}^{h} (y_{pi}^T Q_i y_{pi} + u_{pi}^T R_i u_{pi})] \tag{3}$$

where $y^T Q y + u^T R u$ is the part of the cost function for the standard LQG design, and

$$\sum_{i=1}^{h} (y_{pi}^T Q_i y_{pi} + u_{pi}^T R_i u_{pi})$$

are added sensitivity terms. We seek a controller that minimizes the cost function V_D. Then the sensitivity of the controller to the parameter uncertainy p_i is reduced in increasing the norm of weighting matrices Q_i and R_i.

The nominal LQG controller is obtained by setting Q_i and R_i to zero, and a controller that minimizes sensitivity only (neglecting nominal performance requirements) is obtained by setting Q and R to zero.

Trajectory Sensitivity Matrices for an Example

To construct the trajectory sensitivity matrices A_p, B_p, C_p, D_p, and M_p for the physical system, we deal with an Euler-Bernoulli simply supported beam shown in Fig. 1. This example is used later for the sensitivity-reducing controller design. In this example, we take the following three quantities as the uncertain parameters: 1) mass density of beam (per length) ρ; 2) flexural rigidity of beam EI; and 3) actuator gain K_a.

It is well known that the natural frequency ω_i and mode shape $\Psi_i(r)$ for the ith mode of a simply supported beam are given by

$$\omega_i = \sqrt{\frac{EI}{\rho}} \left(\frac{i\pi}{L}\right)^2, \qquad \Psi_i(r) = \sqrt{\frac{2}{\rho L}} \sin\left(\frac{i\pi}{L} r\right)$$

where L = length of beam.

If we assume a torquer at $r = r_c$, a linear displacement measurement sensor at $r = r_m$, and a linear displacement at $r = r_o$ for the output, then we obtain the following equations of motion:

$$\ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i = b_i(u + w), \qquad i = 1, \ldots, n$$

$$y = \mu(r_o, t) = \sum_{i=1}^{n} \Psi_i(r_o) q_i$$

$$z = \mu(r_m, t) + v = \sum_{i=1}^{n} \Psi_i(r_m) q_i + v$$

where

$$b_i = \Phi_i(r_c) = \frac{\partial}{\partial r} \Psi_i(r)\big|_{r=r_c} = \frac{i\pi}{L} \sqrt{\frac{2}{\rho L}} \cos(i\pi/L) r_c$$

If we choose the state variable x by $x = [q^T, \dot{q}^T]^T$, then the preceding equations can be transformed into the following state-space expression.

$$\dot{x} = Ax + Bu + Dw, \qquad y = Cx, \qquad z = Mx + v$$

where

$$A = \begin{bmatrix} 0 & I \\ -\Omega^2 & -2\zeta\Omega \end{bmatrix}, \quad \Omega = \text{diag } [\omega_1, \ldots, \omega_n], \ \zeta \text{ scalar}$$

$$B = [0 \cdots 0 \ \Phi_1(r_c) \cdots \Phi_n(r_c)]^T \triangleq [0 \quad B_u^T]^T$$

$$C = [\Psi_1(r_o) \cdots 0 \cdots 0] \triangleq [C_y \quad 0]$$

$$D = B$$

$$M = [\Psi_1(r_m) \cdots \Psi_n(r_m) \ 0 \cdots 0] \triangleq [M_z \quad 0]$$

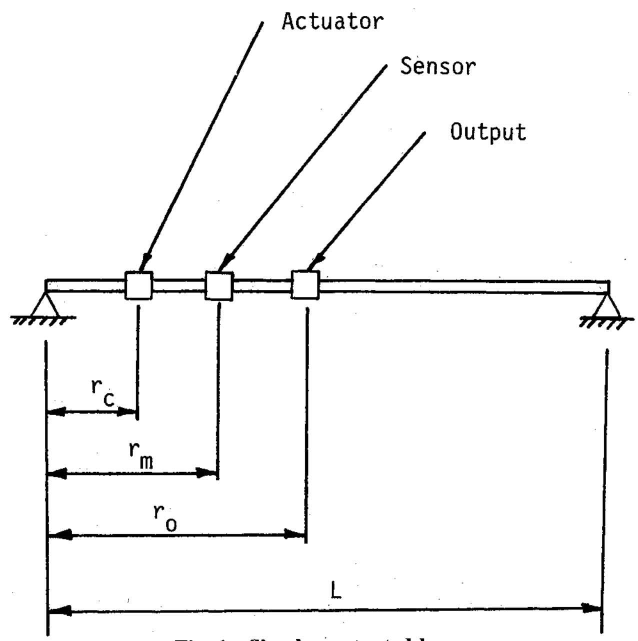

Fig. 1 Simply supported beam.

Using the preceding plant model, we obtain the following trajectory sensitivity matrices A_{pi}, B_{pi}, C_{pi}, D_{pi}, and M_{pi} for the three uncertain physical parameters.

Uncertain parameter ρ:

$$p_1 = \frac{\rho}{(\rho)_{\text{NOM}}}\quad \bar{p}_1 = 1$$

$$A_{p_1} = \begin{bmatrix} 0 & 0 \\ \Omega^2_{\text{NOM}} & \zeta\Omega_{\text{NOM}} \end{bmatrix},\quad B_{p_1} = \begin{bmatrix} 0 \\ -\tfrac{1}{2}(B_u)_{\text{NOM}} \end{bmatrix}$$

$$C_{p_1} = \begin{bmatrix} -\tfrac{1}{2}(C_y)_{\text{NOM}} & 0 \end{bmatrix},\quad D_{p_1} = B_{p_1}$$

$$M_{p_1} = \begin{bmatrix} -\tfrac{1}{2}(M_y)_{\text{NOM}} & 0 \end{bmatrix}$$

Uncertain parameter EI:

$$p_2 = \frac{EI}{(EI)_{\text{NOM}}},\quad \bar{p}_2 = 1$$

$$A_{p_2} = \begin{bmatrix} 0 & 0 \\ -\Omega^2_{\text{NOM}} & -\zeta\Omega_{\text{NOM}} \end{bmatrix}$$

$$B_{p_2} = 0,\quad C_{p_2} = 0,\quad D_{p_2} = 0,\quad M_{p_2} = 0$$

Uncertain parameter K_a:

$$p_3 = \frac{K_a}{(K_a)_{\text{NOM}}},\qquad \bar{p}_3 = 1$$

$$A_{p3} = 0,\qquad B_{p3} = B,\qquad C_{p3} = 0,\qquad D_{p3} = 0,\qquad M_{p3} = 0$$

III. Trajectory Sensitivity Optimization Method

In this section, we introduce our controller design synthesis, which reduces the sensitivity to parameter variations of the plant, using the trajectory sensitivity derived in Sec. II. The basic idea of this method is similar to that of Ref. 2, but the advantages of the new method are 1) the order of the controller (the number of states) obtained by this method is smaller than that of Ref. 2, and 2) the method can deal with a wider class of parameter uncertainty than the method in Ref. 2 (this method cannot deal with the parameter uncertainty related to the measurement matrix, i.e., the M_p term).

The method has the following advantages over the maximum entropy method: It can deal with 1) parameter uncertainty in a nonlinear manner, 2) parameter uncertainty in the disturbance matrix D and the output matrix C, and 3) parameter uncertainty appearing in both the control matrix B and the measurement matrix M at the same time. The advantage of the maximum entropy method is fewer equations to solve. Some discussion of the convergence of the method appears in Ref. 12.

Problem Statement

We consider the following problem of a system described by an nth order plant:

$$\dot{x} = A(p)x + B(p)u + D(p)w \tag{4a}$$

$$y = C(p)x \tag{4b}$$

$$z = M(p)x + v \tag{4c}$$

where

$$E\{w(t)w(\tau)^T\} = W\delta(t-\tau),\quad E\{v(t)v(\tau)^T\} = V\delta(t-\tau)$$

$$E\{w(t)\} = 0,\qquad E\{v(t)\} = 0$$

and $p = (p_1, \ldots, p_h)$ are the uncertain parameters.

The nth order controller is

$$u = Gx_c \tag{5a}$$

$$\dot{x}_c = A^c x_c + B^c u + F(z - M^c x_c)$$

$$= (A^c + B^c G - FM^c)x_c + Fz \tag{5b}$$

where

$$A^c = A(\bar{p}),\qquad B^c = B(\bar{p}),\qquad M^c = M(\bar{p})$$

$\bar{p}$: Nominal value of p

$$\frac{\partial F}{\partial p_i} = 0,\quad \frac{\partial G}{\partial p_i} = 0,\ (i = 1, \ldots, h)$$

Find F and G such that the cost function V_D is minimized.

$$V_D = E_\infty \Bigg\{ y^T Q y + u^T R u + \sum_{i=1}^{h} \beta_i \left[\left(\frac{\partial y}{\partial p_i}\right)^T Q \left(\frac{\partial y}{\partial p_i}\right) + \left(\frac{\partial u}{\partial p_i}\right)^T R \left(\frac{\partial u}{\partial p_i}\right) \right] \Bigg\} \tag{6}$$

If we set $\beta_i = \beta\sigma_i$, where $\sigma_i = |\Delta p_i|$ is the magnitude of the expected variations in p_i, then we need only to determine β, Q, and R as design parameters. The weight β is usually determined through tradeoff between the robustness to parameter variation and the nominal performances of input and output cost. The weights Q and R may be determined under nominal conditions ($\sigma_i = 0$) to satisfy $E_\infty y_i^2 \le \sigma_i^2$, for a specified σ_i, $i = 1, 2, \ldots, n_y$, while minimizing $u^T R u$. The algorithm for such weights is given in Chap. 8 of Ref. 10.

Derivation of Necessary Conditions

Let

$$\tilde{x} = x - x_c \tag{7}$$

then the equations for the closed-loop system are given by

$$\dot{\tilde{x}} = (A - FM)\tilde{x} + \{(A + BG - FM) - (A^c + B^c G - FM^c)\}x_c + Dw - Fv \tag{8}$$

$$\dot{x} = FM\tilde{x} + \{A^c + B_c G + F(M - M^c)\}x_c + Fv \tag{9}$$

$$y = C\tilde{x} + Cx_c \tag{10}$$

$$u = Gx_c \tag{11}$$

The preceding equations are transformed into the following matrix forms:

$$\dot{x}_a = A_a x_a + D_a w_a \tag{12a}$$

$$y = C_a x_a \tag{12b}$$

$$u = G_a x_a \tag{12c}$$

where

$$x_a = \begin{Bmatrix} \tilde{x} \\ x_c \end{Bmatrix},\qquad w_a = \begin{Bmatrix} w \\ v \end{Bmatrix}$$

$$A_a = \begin{bmatrix} A - FM & (A + BG - FM) - (A^c + B^c G - FMc) \\ FM & A^c + B^c G + F(M - M^c) \end{bmatrix}$$

$$D_a = \begin{bmatrix} D & -F \\ 0 & F \end{bmatrix},\qquad C_a = [C \quad C],\qquad G_a = [0 \quad G]$$

Let

$$x_s = [\tilde{x}^T\ x_c^T\ \tilde{x}_{p1}^T\ x_{c_{p1}}, \ldots, \tilde{x}_{ph}^T\ x_{c_{ph}}^T]^T \tag{13a}$$

$$y_s = [y^T\ u^T\ y_{p1}^T\ u_{p1}^T, \ldots, y_{ph}^T\ u_{ph}^T]^T \tag{13b}$$

$$w_s = [w^T\ v^T]^T \tag{13c}$$

then the closed-loop sensitivity system is expressed by

$$\dot{x}_s = A_s x_s + D_s w_s \tag{14a}$$

$$y_s = C_s x_s \tag{14b}$$

where

$$A_s = \begin{bmatrix} A_0 & 0 & 0 & \ldots & 0 \\ A_1 & A_0 & 0 & \ldots & 0 \\ \vdots & 0 & A_0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ A_h & 0 & 0 & \ldots & A_0 \end{bmatrix} \tag{14c}$$

$$A_0 = A_a(\bar{p}) = \begin{bmatrix} A(\bar{p}) - FM(\bar{p}) & 0 \\ FM(\bar{p}) & A(\bar{p}) + B(\bar{p})G \end{bmatrix} \tag{14d}$$

$$A_i = \left.\frac{\partial(A_a)}{\partial p_i}\right|_{p=\bar{p}} = \begin{bmatrix} A_{pi}(\bar{p}) - FM_{pi}(\bar{p}) & A_{pi}(\bar{p}) + B_{pi}(\bar{p})G - FM_{pi}(\bar{p}) \\ FM_{pi}(\bar{p}) & FM_{pi}(\bar{p}) \end{bmatrix}$$

$$i = 1, \ldots, h \tag{14e}$$

$$D_s = \begin{bmatrix} D & -F \\ 0 & F \\ D_{p1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ D_{ph} & 0 \\ 0 & 0 \end{bmatrix} \tag{14f}$$

$$C_s = \begin{bmatrix} C & C & 0 & 0 & \ldots & 0 & 0 \\ 0 & G & 0 & 0 & \ldots & 0 & 0 \\ C_{p1} & C_{p1} & C & C & \ldots & 0 & 0 \\ 0 & 0 & 0 & G & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ C_{ph} & C_{ph} & 0 & 0 & \ldots & C & C \\ 0 & 0 & 0 & 0 & \ldots & 0 & G \end{bmatrix} \tag{14g}$$

From the triangular structure of A_s in Eq. (14c), note that the poles of the closed-loop sensitivity system given by Eq. (14) are equal to those of the closed-loop system without sensitivity states repeated $(h+1)$ times. Therefore, if the closed-loop system is stable at $p = \bar{p}$, then the closed-loop sensitivity system given by Eq. (14) is always stable.

The state-steady covariance is defined by

$$X_s = E_\infty\{x_s x_s^T\}$$

$$= E_\infty \begin{bmatrix} \tilde{x}\tilde{x}^T & \tilde{x}x_c^T & \tilde{x}\tilde{x}_{p1}^T & \tilde{x}x_{c_{p1}}^T & \ldots & \tilde{x}\tilde{x}_{ph}^T & \tilde{x}_{ph}x_{c_{ph}}^T \\ x_c x^T & x_c x_c^T & x_c x_{p1}^T & x_c x_{c_{p1}}^T & \ldots & x_c\tilde{x}_{ph}^T & x_c x_{c_{ph}}^T \\ & & & & & \vdots & \vdots \\ & & & & & \tilde{x}_{ph}\tilde{x}_{ph}^T & \tilde{x}_{ph}x_{c_{ph}}^T \\ \mathrm{S.} & \mathrm{Y.} & \mathrm{M.} & & & x_{c_{ph}}\tilde{x}_{ph}^T & x_{c_{ph}}x_{c_{ph}}^T \end{bmatrix} \tag{15a}$$

Then X_s is obtained as the solution of the following Lyapunov equation:

$$A_s X_s + X_s A_s^T + D_s W_s D_s^T = 0 \tag{15b}$$

where

$$W_s = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \tag{16}$$

Using X_s, we can express the cost function V_D by

$$V_D = \mathrm{tr}\ [X_s C_s^T Q_s C_s] \tag{17}$$

where

$$Q_s = \text{block diag } (Q,\ R,\ \beta_1 Q,\ \beta_1 R, \ldots, \beta_h,\ Q,\ \beta_h R) \tag{18}$$

By augmenting the constraints in Eq. (15b) to the objective functions of Eq. (17) by use of Lagrange multipliers, we introduce H given by

$$H = \mathrm{tr}\ [X_s C_s^T Q_s C_s] + \mathrm{tr}\ [K_s(A_s X_s + X_s A_s^T + D_s W_s D_s^T)] \tag{19}$$

Then the solution to the problem satisfies the following conditions:

$$\frac{\partial H}{\partial X_s} = 0, \quad \frac{\partial H}{\partial K_s} = 0, \quad \frac{\partial H}{\partial G} = 0, \quad \frac{\partial H}{\partial F} = 0 \tag{20}$$

Relying on standard matrix calculus, the following necessary conditions are derived:

$$\frac{\partial H}{\partial X_s} = 0$$

$$K_s A_s + A_s^T K_s + C_s^T Q_s C_s = 0 \tag{21}$$

$$\frac{\partial H}{\partial K_s} = 0$$

$$A_s X_s + X_s A_s^T + D_s W_s D_s^T = 0 \tag{22}$$

$$\frac{\partial H}{\partial G} = 0$$

$$G = -R^{-1}\{B^T \sum_{k=1}^{h+1} K_{2k} X_{2k} + \sum_{k=1}^{h} B_{pk}^T (K_{2k+1} X_2)\} \times [X_{22} + \sum_{k=1}^{h} \beta_k X_{2k+1,2k+1}]^{-1} \tag{23}$$

$$\frac{\partial H}{\partial F} = 0$$

$$F = -(K_{11} - K_{12} - K_{12}^T + K_{22})^{-1}\ [\{\sum_{k=1}^{h+1} (K_{2k} - K_{2k-1}) X_{2k-1}\} \times M^T + \sum_{k=1}^{h} (K_{2k+2} - K_{2k+1})(X_1 + X_2) M_{pk}^T]\ V^{-1} \tag{24}$$

where $K_1, \ldots, K_{2h+2}$ and $X_1, \ldots, X_{2h+2}$ are defined by

$$K_s = [K_1^T K_2^T \cdots K_{2h+2}^T]^T \qquad K_i: n \times 2n(h+1) \tag{25}$$

$$X_s = [X_1\ X_2 \cdots X_{2h+2}] \qquad X_i: 2n(h+1) \times n \tag{26}$$

X_{ij} and K_{ij} are the (i,j) block of X_s and K_s.

$$X_s = \begin{bmatrix} X_{1,1} & \cdots & X_{1,2h+2} \\ \vdots & \cdots & \vdots \\ X_{2h+2,1} & \cdots & X_{2h+2,2h+2} \end{bmatrix} \quad X_{ij}: n \times n \tag{27}$$

$$K_s = \begin{bmatrix} K_{1,1} & \cdots & K_{1,2h+2} \\ \vdots & \cdots & \vdots \\ K_{2h+2,1} & \cdots & K_{2h+2,2h+2} \end{bmatrix} \quad K_{ij}: n \times n \tag{28}$$

We can see in the definition of A_s that it contains F and G. Therefore, Eqs. (21-24) are coupled. However, these equations can be solved numerically by using the iterative method presented later in this section. Before proceeding to the iterative method, we investigate the Lyapunov equations given by Eqs. (21) and (22).

The size of the Lyapunov equations is $2n(1+h) \times 2n(1+h)$. Therefore, if the number of uncertain parameters is high, the equations may be too large to solve directly. The structure of the equations, however, allows them to be solved in partitioned forms.

Partitioning of the Lyapunov Equation for X_s

Let Z_{ij} represent the $2n \times 2n$ matrix

$$Z_{ij} \triangleq \begin{bmatrix} X_{2i-1,2j-1} & X_{2i-1,2j} \\ X_{2i,2j-1} & X_{2i,2j} \end{bmatrix} \tag{29}$$

and D_{ij} be the $2n \times 2n$ matrices given by

$$D_{11} \triangleq \begin{bmatrix} DWD^T + FVF^T & -FVF^T \\ -FVF^T & FVF^T \end{bmatrix} \tag{30}$$

$$D_{ij} \triangleq \begin{bmatrix} D_{p_{i-1}} W D_{p_{j-1}}^T & 0 \\ 0 & 0 \end{bmatrix}, \quad i \geq 1, j \geq 2 \text{ and } j \geq i,\ D_{p_o} \triangleq D \tag{31}$$

then Z_{ij} is the $2n \times 2n$ block of X_s in the ij position, and D_{ij} is the $2n \times 2n$ block of $D_s W_s D_s^T$ in the ij position. The partitioned matrices Z_{ij} $(j \geq i \geq 1)$ are obtained by solving the following equations:

(1,1) Block:

$$A_0 Z_{11} + Z_{11} A_0^T + D_{11} = 0 \tag{32a}$$

(1,j) Block:

$$A_0 Z_{1j} + Z_{1j} A_0^T + Z_{11} A_{j-1}^T + D_{1j} = 0, \qquad j = 2, \ldots, h+1 \tag{32b}$$

(i,j) Block:

$$A_0 Z_{ij} + Z_{ij} A_0^T + (A_{i-1} Z_{1j} + Z_{1j}^T A_{j-1}^T) + D_{ij} = 0 \qquad 2 \leq i \leq j \leq h+1 \tag{32c}$$

Since the (1,1) block element given by Eq. (32a) is the standard Lyapunov equation, it may be solved directly for Z_{11}. Substitution of Z_{11} obtained in Eq. (32a) into the (1,j) block elements reduces Eq. (32b) to standard Lyapunov equations. In a similar way, the (i,j) block $(2 \leq i \leq j \leq h+1)$ elements can be solved by substituting Z_{1i} and Z_{1j} obtained in the previous calculations.

Therefore, the Lyapunov equation for X_s of order $2n(1+h)$ can be reduced to the $(h+1)(h+2)/2$ Lyapunov equations of order $2n$. It can also be shown easily that each partitioned Lyapunov equation obtained previously can be partitioned further into four subblock elements of order n, which can be solved separately in the sequence of (1,1), (1,2), (2,1), (2,2) subblock. Hence, the total number of Lyapunov equations of order n becomes $2(h+1)(h+2)$. However, from the symmetric property of X_s, the diagonal block elements Z_{ii} $(i = 1, \ldots, h+1)$ are reduced to three subblock elements of order n instead of four subblocks elements. Therefore, the total number of Lyapunov equations to be solved is $(h+1)(2h+3)$.

Partitioning of the Lyapunov Equation for K_s

Let Y_{ij} represent the $2n \times 2n$ matrix

$$Y_{ij} \triangleq \begin{bmatrix} K_{2i-1,2j-1} & K_{2i-1,2j} \\ K_{2i,2j-1} & K_{2i,2j} \end{bmatrix} \tag{33}$$

and C_{ij} be the $2n \times 2n$ matrices given by

$$C_{11} \triangleq \begin{bmatrix} C^TQC + \sum_{k=1}^{h} C_{p_k}^T Q_k C_{p_k} & C^TQC + \sum_{k=1}^{h} C_{p_k}^T Q_k C_{p_k} \\ C^TQC + \sum_{k=1}^{h} C_{p_k}^T Q_k C_{p_k} & C^TQC + \sum_{k=1}^{h} C_{p_k}^T Q_k C_{p_k} + G^TRG \end{bmatrix} \tag{34a}$$

$$C_{1j} \triangleq \begin{bmatrix} C_{p_{j-1}}^T Q_{j-1} C & C_{p_{j-1}}^T Q_{j-1} C \\ C_{p_{j-1}}^T Q_{j-1} C & C_{p_{j-1}}^T Q_{j-1} C \end{bmatrix}, \quad 2 \leq j \leq h+1 \tag{34b}$$

$$C_{ii} \triangleq \begin{bmatrix} C^TQ_{i-1}C & C^TQ_{i-1}C \\ C^TQ_{i-1}C & C^TQ_{i-1}C + G^TR_{i-1}G \end{bmatrix}, \quad 2 \leq i \leq h+1 \tag{34c}$$

$$C_{ij} \triangleq 0, \quad 2 \leq i < j \tag{34d}$$

where $Q_R = \beta_k Q$ and $R_k = \beta_k R$. Then Y_{ij} is the $2n \times 2n$ block of K_s in the ij position. The partitioned matrices Y_{ij} $(j \geq i \geq 1)$ position, and C_{ij} is the $2n \times 2n$ block of $C_s^T Q_s C_s$ in the ij are obtained by solving the following equations:

(j,j) Block:

$$Y_{jj} A_0 + A_0^T Y_{jj} + C_{jj} = 0, \qquad 1 < j \leq h+1 \tag{35a}$$

(i,j) Block:

$$Y_{ij} A_0 + A_0^T Y_{ij} + C_{ij} = 0, \qquad 1 < i < j \tag{35b}$$

(1,j) Block:

$$Y_{1j} A_0 + A_0^T Y_{1j} + A_{j-1}^T Y_{jj} + C_{1j} = 0, \qquad 2 < j \tag{35c}$$

(1,1) Block:

$$Y_{11} A_0 + A_0^T Y_{11} + (Y_{12} A_1 + Y_{13} A_2 + \cdots + Y_{1,h+1} A_h) + (Y_{12} A_1 + Y_{13} A_2 + \cdots + Y_{1,h+1} A_h)^T + C_{11} = 0 \tag{35d}$$

From Eq. (35) and $C_{ij} = 0$ $(2 \le i \le j)$, we obtain

$$Y_{ij} = 0, \qquad \text{for} \qquad 1 < i < j \tag{36}$$

Therefore, the structure of K_s is given by

$$K_s = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & \cdots & Y_{1,h+1} \\ & \beta_1 Y_0 & 0 & \cdots & 0 \\ & & \beta_2 Y_0 & \cdots & \vdots \\ & & & \cdots & 0 \\ \text{S.} & \text{Y.} & \text{M.} & & \beta_h Y_0 \end{bmatrix} \tag{37}$$

where Y_0 is the solution for the following Lyapunov equations:

$$Y_0 A_0 + A_0^T Y_0 + C_q = 0 \tag{38}$$

$$C_q = \begin{bmatrix} C^TQC & C^TQC \\ C^TQC & C^TQC + G^TRG \end{bmatrix} \tag{39}$$

The number of Lyapunov equations to be solved becomes $h + 2$ Lyapunov equations of order $2n$ or $4h + 6$ Lyapunov equations of order n. When we choose $\beta_1 = \beta_2 = \cdots = \beta_h = 0$, only the (1,1) block K_s becomes nonzero. In this case, it can be easily shown that the equations for G and F reduce to the standard LQG equations.

Algorithm to Obtain the Numerical Solution

Since the equations obtained as the necessary conditions are coupled, a special numerical algorithm is required. The approach taken is similar to that of Refs. 2 and 13. The algorithm is summarized as follows.

Algorithm to Obtain F and G

Step 1. Choose initial F and G

$$F = F_0, \qquad G = G_0$$

(e.g., use the solution for the standard LQG problem).

Step 2. For the given F and G, solve X_s and K_s from Eq. (21) and Eq. (22) using the partitioned Lyapunov equations.

Step 3. Using X_s and K_s obtained in step 2, calculate F and G. Set $F_{\text{NEW}} = F$ $G_{\text{NEW}} = G$.

Step 4. If $\|G_{\text{NEW}} - G_{\text{OLD}}\| < \epsilon_g$ and $\|F_{\text{NEW}} - F_{\text{OLD}}\| < \epsilon_f$, then stop: solution completed. Otherwise, set

$$F = F_{\text{NEW}}\alpha + F_{\text{OLD}}(1-\alpha)$$

$$G = G_{\text{NEW}}\alpha + G_{\text{OLD}}(1-\alpha)$$

Return to Step 2.

The coefficient that dictates the convergence of the solution is α. Usually, 0.2–0.5 is used as the value of α. But when the weight β_i to the sensitivity part of the cost is large, a smaller value may be necessary.

IV. Numerical Example and Performance Comparison with other Controller Design Methods

In order to investigate the effectiveness of the proposed controller design synthesis and compare the performances with other methods, we take the following three examples.

Simply Supported Beam

We have already derived the sensitivity trajectory model for a simply supported beam in Sec. II. Here we consider the same example. For the numerical values, we use the following:

Beam parameter:

$$L = \pi, \qquad \rho = \frac{2}{L} \; EI = \rho$$

Sensor, actuator, output position:

$$r_m = 0.30\,L, \qquad r_c = 0, \qquad r_0 = 0.45\,L$$

Noises:

$$V = 1, \qquad W = 1$$

Then ω_i, $\Psi_i(r)$, and $\Phi_i(r)$ are given by

$$\omega_i = i^2, \qquad \Psi_i(r) = \sin\left(i\frac{\pi}{L}r\right), \qquad \Phi_i(r) = i\cos\left(i\frac{\pi}{L}r\right),$$

$$i = 1, \cdots, n$$

If we choose the first 4 modes as our design and evaluation model, then we obtain

$$\Omega = \text{diag}\,(1^2, 2^2, 3^2, 4^2), \qquad B_u = [1\;2\;3\;4]^T$$

$$C_y = [\sin(0.45\pi)\;\sin(2\times0.45\pi)\;\sin(3\times0.45\pi)\;\sin(4\times0.45\pi)]$$

$$M_z = [\sin(0.30\pi)\;\sin(2\times0.30\pi)\;\sin(3\times0.30\pi)\;\sin(4\times0.30\pi)]$$

Substituting these matrices into A, B, C, D, M, and A_{pi}, B_{pi}, C_{pi}, D_{pi}, M_{pi}, we obtain the required data for the trajectory sensitivity optimization (TSO) method design synthesis.

If we choose $Q = 1$ and $R = 1$ for the weights of the cost function and apply the standard LQG method, then we obtain the controller whose stability range for parameter variation and input and output cost are summarized in Table 1. As we can see from Table 1, the standard LQG controller is sensitive to the parameter variation ρ and EI.

We apply the TSO to the same system to reduce the sensitivity. First, we investigate one uncertain parameter case in which only one uncertain parameter is considered for the design of

Table 1 Performances of the standard LQG controller

Standard LQG controller		Performances
Gain margin	$K_{\max}$	4.087
Stability range of ρ	$P_{\max}$	1.173
$P = \rho/(\rho)_{\text{NOM}}$	$P_{\min}$	0.677
Stability range of EI	$P_{\max}$	3.226
$P = EI/(EI)_{\text{NOM}}$	$P_{\min}$	0.863
Input and outut cost	V_y	2.760
	V_u	0.488

Table 2 Performances of controllers type A

Sensitivity reducing controller for ρ type A		Performances		
		Type A-1 $\beta = 0.01$	Type A-2 $\beta = 0.1$	Type A-3 $\beta = 0.5$
Gain margin	$K_{\max}$	3.247	2.300	1.706
Stability range ρ	$P_{\max}$	1.415	1.687	1.802
$P = \rho/(\rho)_{\text{NOM}}$	$P_{\min}$	0.681	0.596	0.435
Stability range of EI	$P_{\max}$	2.774	2.704	2.734
$P = EI/(EI)_{\text{NOM}}$	$P_{\min}$	0.733	0.631	0.607
Input and output cost	V_y	2.856	3.174	3.821
	V_u	1.300	3.709	11.243

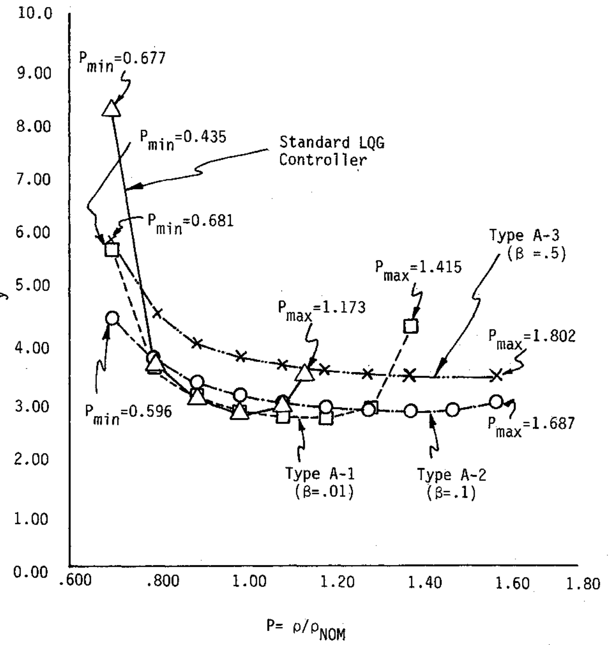

Fig. 2 Output cost performance change due to ρ variation for different controllers.

Table 3 Performances of controllers type B

Sensitivity-reducing controllers for ρ and K_a type B		Type A-2 $\beta_1=0.1$ $\beta_2=0$	Type B-1 $\beta_1=0.1$ $\beta_2=1$	Type B-2 $\beta_1=0.1$ $\beta_2=10$
		Performances		
Gain margin	K_{max}	2.300	3.054	4.843
Stability range ρ	P_{max}	1.687	1.567	1.360
$P=\rho/(\rho)_{NOM}$	P_{min}	0.596	0.661	0.687
Stability range of EI	P_{max}	2.704	1.618	1.526
$P=EI/(EI)_{NOM}$	P_{min}	0.631	0.669	0.755
Input and output cost	V_y	3.147	2.913	2.723
	V_u	3.709	3.692	3.865

the sensitivity-reducing controller. Next, we deal with the two-uncertain-parameter case in which two uncertain parameters are considered at the same time for the controller design.

Sensitivity-Reducing Controllers for the One-Uncertain-Parameter Case: Controller Type A

In this case, the sensitivity part with respect to mass density variation is weighted for the TSO cost. Table 2 shows the stability range for parameter variation and input and output cost of type A controllers for different weights β. When we compare Table 1 and Table 2, we see that the stability range for ρ variation increases as the weight β increases. In this case, the sensitivity to EI variation is also reduced.

The gain margin, however, decreases as the weight β increases. Figure 2 shows output cost performance change due to ρ variation for different controllers. As we see in Fig. 2 and Table 2, the output cost increase rate is maintained relatively small while the input cost increases pretty rapidly as β increases. Therefore, tradeoff between the robustness to parameter variation and the input and output cost should be made to determine the appropriate weight β.

Sensitivity-Reducing Controllers for the Two-Uncertain-Parameter Case: Controller Type B

In this case, two sensitivity terms (i.e., sensitivity terms with respect to ρ and K_a) are weighted at the same time for the TSO cost. Table 3 shows the performance of type B controllers. Comparing the results with the standard LQG controller performance, we notice that controller B-2 can achieve better robustness for parameter variations ρ, EI, and K_a also.

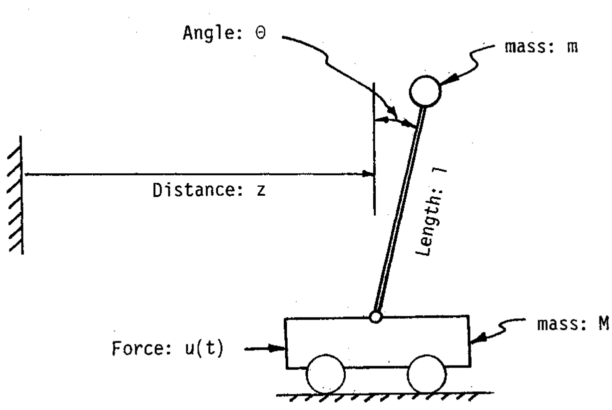

Fig. 3 Cart with an inverted pendullum.

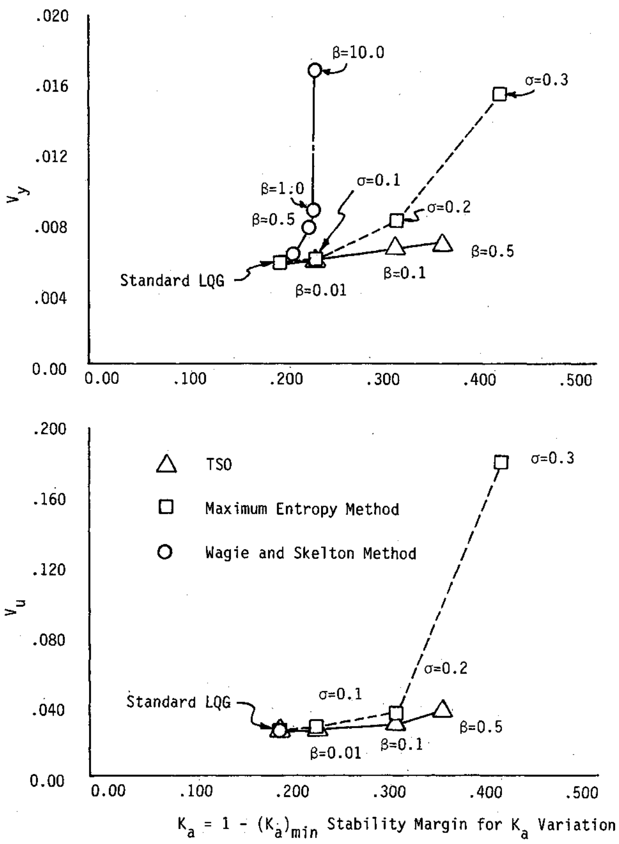

Fig. 4 Performance comparison of different controller designs (uncertain parameter: K_a).

Cart with an Inverted Pendulum

Next, we consider the cart with an inverted pendulum shown in Fig. 3. The linearized equations expressed in state-variables form are given by

$$\frac{d}{dt}\begin{Bmatrix} z \\ \dot{z} \\ \theta \\ \dot{\theta} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{M+m}{Ml}g & 0 \end{bmatrix} \begin{Bmatrix} z \\ \dot{z} \\ \theta \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ -\dfrac{1}{Ml} \end{bmatrix} u$$

or

$$\dot{x} = Ax + B(u + w)$$

$$y = z = [1 \quad 0 \quad 0 \quad 0]\, x = Cx$$

$$z_m = \begin{Bmatrix} z \\ \theta \end{Bmatrix} + \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x + v = Mx + v$$

We consider the following two uncertain parameters:

1) *Actuator Gain* (K_a): We obtain the following sensitivity data.

Trajectory sensitivity data:

$$p = \frac{K_a}{(K_a)_{\rm NOM}}$$

$$A_p = 0, \quad B_p = B, \quad C_p = 0, \quad D_p = 0, \quad M_p = 0$$

Maximum entropy design data:

$$A_1 = 0, \quad B_1 = 0, \quad M_1 = 0$$

2) *Angle Sensor Gain* (K_s): We obtain the following sensitivity data.

Trajectory sensitivity data:

$$p = \frac{K_s}{(K_s)_{\rm NOM}}$$

$$A_p = 0, \quad B_p = 0, \quad C_p = 0, \quad D_p = 0, \quad M_p = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Maximum entropy design data:

$$A_1 = 0, \quad B_1 = 0, \quad M_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Numerical values:

$$M = 1 \text{ kg}, \quad m = 0.1 \text{ kg}, \quad L = 1.0 \text{ m}, \quad g = 9.8 \text{ m/s}^2$$

$$W = 1.0 \times 10^{-6}, \quad V = 1.0 \times 10^{-6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 1 = Q$$

Sensitivity-Reducing Controllers for K_a Variation

Three different sensitivity-reducing controller design syntheses, 1) TSO, 2) Wagie and Skelton method, and 3) maximum entropy method, were applied to this problem to compare the performance. The performance curves (input and output cost vs stability range of K_a variation) for the different controller designs are shown in Fig. 4. As shown in the figure, the Wagie and Skelton method cannot improve the robustness for K_a variation even if a large β is chosen. The TSO achieves smaller input and output cost than the maximum entropy method for the same stability margin. Therefore, the TSO is best for this problem in terms of performance cost and robustness to parameter variation.

Sensitivity-Reducing Controllers for K_s Variation

The Wagie and Skelton method cannot deal with the problem in which the measurement sensitivity matrix M_p is nonzero. Therefore, the TSO and the maximum entropy methods were applied to this problem. The results are shown in Fig. 5. In this case, the TSO achieves smaller output cost than the maximum entropy method for the same stability margin, whereas it requires a larger input cost than the maximum

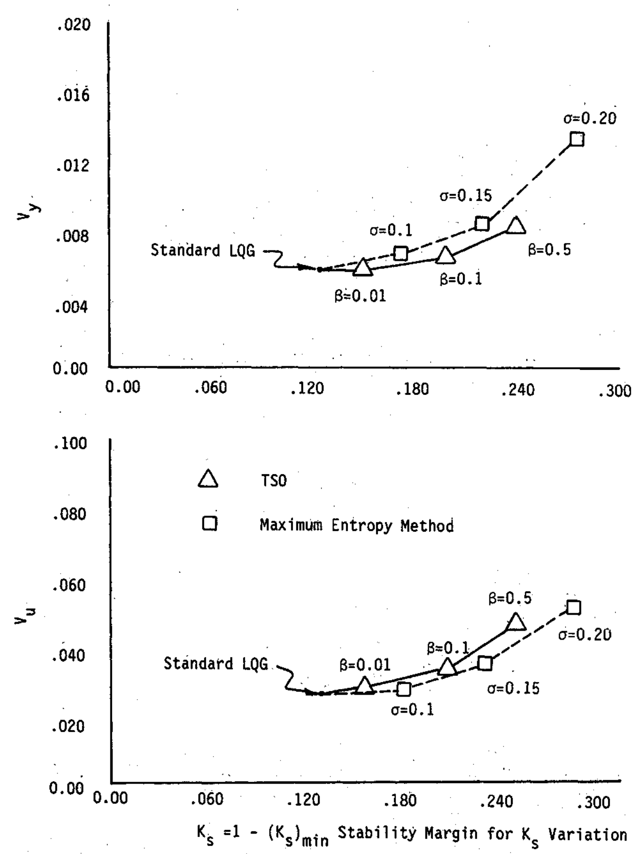

Fig. 5 Performance comparison of different controller designs (uncertain parameter: K_s).

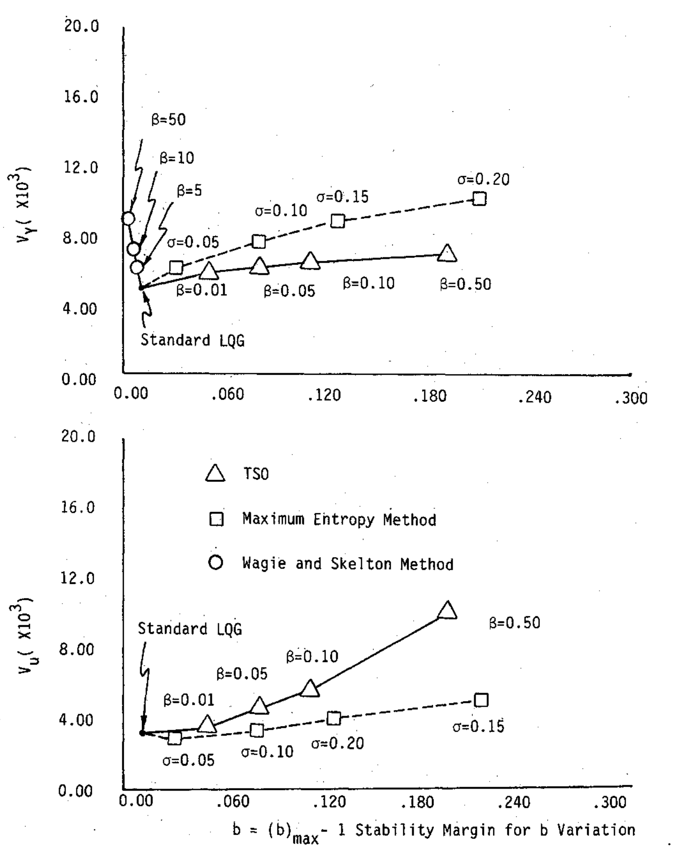

Fig. 6 Performance comparison of different controller designs (Doyle's example).

entropy method. Figure 5 suggests that in this case there is no big difference between the two methods with respect to performance cost and stability margin for K_a variation.

Doyle's Example

The problem considered here was first given by Doyle[1] and investigated further by Bernstein in his maximum entropy method.[7] The required data for the problem are given by

$$A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}0\\ b\end{bmatrix},\quad C=[1\quad 1],\ D=\begin{bmatrix}1\\ 1\end{bmatrix},\quad M=[1\quad 0]$$

Uncertain parameter: b in matrix B $(b)_{\text{NOM}}=1$

$$W=60.0,\quad V=1.0,\quad R=1.0,\quad Q=60.0$$

As in the previous example, three different controller design syntheses were applied to this problem. The performance curves for the different controller designs are shown in Fig. 6. In this case, the controllers obtained by the Wagie and Skelton method show worse robustness for parameter variation b than the standard LQG controller. Therefore, the method is unacceptable for this problem. The TSO and the maximum entropy method[7] show similar results to those obtained in the previous example (uncertain parameter: K_s as in Fig. 5).

V. Conclusions

A new controller design synthesis is presented to improve sensitivity to parameter uncertainty. The proposed method uses the trajectory sensitivity to model the parameter uncertainty and introduces a special cost function to reduce the parameter sensitivity at both the input and output to the plant. The order of the controller is equal to that of the nominal plant as opposed to the high-order sensitivity controllers previously available. The necessary conditions for the optimization consist of two Lyapunov equations and two gain-matrix equations. An iterative algorithm obtains the solution to these coupled equations. The large sizer of the Lyapunov equations of order $2n(h+1)$ reduces to several smaller equations, using partitioned forms improving numerical efficiency.

This new method deals with a wider class of parameter uncertainty than any other method. It deals with parameters appearing nonlinearly in any place: the plant matrix, the input matrix, the disturbance matrix, the output matrix, and the measurement matrix.

Numerical examples show that the method is effective in improving robustness to parameter variations. The disadvantage of the method is the lack of a closed-form solution. Iterative algorithms, whose convergence remains an open question, are required for future research.

References

[1]Doyle, J. C., "Guaranteed Margins for LQG Regulators," *IEEE Transactions on Automatic Control*, Vol. 5C-23, 1978, pp. 756–757.

[2]Wagie, D. A., and Skelton, R. E., "Model Reduction and Controller Synthesis in the Presence of Parameter Uncertainty," *Automatica*, Vol. 22, 1986, pp. 295–308.

[3]Bernstein, D. S., and Hyland, D. C., "The Optimal Projection/Maximum Entropy Approach to Designing Low-Order, Robust Controllers for Flexible Structures," *Proceedings of the 24th IEEE CDC*, Dec. 1985, pp. 745–752.

[4]Tahk, M., and Speyer, J., "Modeling of Parameter Variations and Asymptotic LQG Synthesis," *IEEE Transactions on Automatic Control*, Vol. AC-32, Sept. 1987, pp. 793–801.

[5]Doyle, J. C., and Stein, G., "Robustness with Observers," *IEEE Transactions on Automatic Control*, Vol. AC-24, Aug. 1979, pp. 607–611.

[6]Skelton, R. E., and Owens, D. H., (eds.), *Model Error Concepts and Compensation*, Pergamon, New York, 1986, 13–20.

[7]Bernstein, D. S., and Greeley, S. W., "Robust Controller Synthesis Using the Maximum Entropy Design Equations," *IEEE Transactions on Automatic Control*, Vol. AC-31, April 1986, pp. 362–365.

[8]Bernstein, D. S., Davis, L. D., Greeley, S. W., and Hyland, D. C., "Numerical Solution of the Optimal Projection/Maximum Entropy Design Equations for Low-Order, Robust Controller Design," *Proceedings of the 24th IEEE CDC*, Dec. 1985, pp. 1795–1798.

[9]Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley, New York, 1972, Chaps. 3,4.

[10]Skelton, R. E., *Dynamic Systems Control*, Wiley, New York, 1988, Chap. 9.

[11]Fortmann, T. E., and Hitz, K. L., *An Introduction Linear Control Systems*, Marcel Dekker, New York, 1977, Chap. 13.

[12]Mil'stein, G. N., "Design of Stabilizing Controller with Incomplete State Data for Linear Stochastic Systems with Multiplicative Noise," *Automatic Remote Control*, Vol. 43, 1982, pp. 653–659.

[13]Yedavalli, R. K., and Skelton, R. E., "Controller Design for Parameter Sensitivity Reduction in Linear Regulators," *Optimal Control Applications and Methods*, Vol. 3, No. 3, 1982, pp. 221–240.